



TITLE:

# SHADOW PRICES FOR PUBLIC INVESTMENT CRITERIA

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CITATION:

Asanuma, Banri. SHADOW PRICES FOR PUBLIC INVESTMENT CRITERIA.  
Kyoto University Economic Review 1969, 39(2): 62-80

ISSUE DATE:

1969-10

URL:

[https://doi.org/10.11179/ker1926.39.2\\_62](https://doi.org/10.11179/ker1926.39.2_62)

RIGHT:

# THE KYOTO UNIVERSITY ECONOMIC REVIEW

MEMOIRS OF THE FACULTY OF ECONOMICS  
IN THE KYOTO UNIVERSITY

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Vol. XXXIX, No. 2

OCTOBER 1969

Whole No. 87

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PUBLISHED BY

THE FACULTY OF ECONOMICS, KYOTO UNIVERSITY

SAKYO-KU, KYOTO, JAPAN

# SHADOW PRICES FOR PUBLIC INVESTMENT CRITERIA

By Banri ASANUMA\*

## Introduction

A variety of discussions about how to construct investment criteria have appeared since the last half of the 1950's, partly out of experiences in making public investments both in advanced and developing countries, and partly out of the development of the theory of business finance<sup>1)</sup>. This problem is nothing other than an application of the theory of intertemporal resource allocation, so that it has a close connection with the traditional marginal analysis of capital and mathematical programming.

In this paper, limiting the scope to the field of public investment, we shall focus our attention on the problem of the discount rate after a brief discussion about investment criteria. Then, using the idea of Fisherian (and Hirshleiferian) marginal analysis as a stepping-stone, we shall attempt to grasp the social rate of discount as a shadow price of a nonlinear program. In the last part of the paper, we shall give a brief comment on problems requiring further consideration.

## I General Form of Investment Criteria

To begin with, let us first observe the form of investment criteria broadly accepted, and look at the relations between them and the theory of optimization. We shall make a distinction between the problem of the evaluation of individual projects and the problem of selecting a suitable set from among alternative projects, and then take up these two problems in that order.

### I. 1. Evaluation of a Project

#### I. 1. 1. Initial Data

From the economist's point of view a project is characterized by a

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1) In the U. S. A., economists at Harvard University have been tackling the problem of water resources development and Eckstein, [10] appeared in 1958 as one of the earliest crystallizations of such study. In France, where nationalization made progress after World War II, studies on investment criteria have been pursued by the public electric power corporation as well as in many other national industries and since about 1960 British and American researchers have begun to turn their eyes toward these works. The work by Massé, [19] reflects the practices in France, and the work by Solomon, [20] collected various arguments about investment criteria developed in the field of theory of business finance in the United States.

stream of revenues and outlays which begins with its starting point and continues until its horizon, or, in the language of the theory of public finance, the stream of its benefits and costs. In the following analysis we shall adopt the period analysis. Let the present moment be expressed as  $t=0$ , and the moment  $\tau$  years later as  $t=\tau$  ( $\tau=1, \dots, T$ ). Let the year from  $t=\tau-1$  to  $t=\tau$  be called the  $\tau$ -th period. Let  $B_\tau$  denote the benefit from the project during the  $\tau$ -th period,  $O_\tau$  the operating cost, and  $K_\tau$  the capital cost [expenditure]. Then, taking  $B_\tau - O_\tau - K_\tau = R_\tau$ , a project can be expressed by a sequence or vector  $(R_0, R_1, \dots, R_T)$ , of which the component  $R_\tau$  ( $\tau=0, 1, \dots, T$ ) is the net benefit in the  $\tau$ -th period.

While  $K_\tau$  in normal cases is thought to accrue concentratedly at the initial moment (and also at the time of renewal),  $B_\tau$ 's and  $O_\tau$ 's are such items that are expected to accrue every year, frequently for 50 years to come and sometimes for 100 years to come, making it a hard task to estimate them.

Furthermore, speaking of the benefit of public investment, so-called indirect benefits and intangible benefits being of great importance, the determination of its extent and evaluation in money terms involve considerable difficulty<sup>2)</sup>. But in this paper we shall not discuss these points. We shall assume that data about the stream of  $B_\tau$ 's,  $O_\tau$ 's and  $K_\tau$ 's are available and that we can disregard uncertainty.

Now, as we can readily see by such an example as two sequences  $(-1, 0, 4)$  and  $(-1, 2, 1)$ , it is generally impossible to determine the relative merit of one project to another by a direct comparison of the sequences of the net benefits. It is necessary to construct an index which can express the economic worth of a project by a single number. For this purpose the following methods have been worked out and are in use:

- (I) The net present value method;
- (II) The benefit-cost ratio method;
- (III) The internal rate of return method;
- (IV) The annual cost method;
- (V) The payback period method.

Here we shall simply take up the net present value method without going into further comparative discussions about these methods<sup>3)</sup>.

#### I. 1. 2. Net Present Value

*Discounting* is the operation of converting economic quantities of different times into commensurable quantities by giving them certain relative weights. The numerical value obtained by adding each term of the net

2) Kumagai, [13], pp. 290, 293.

3) For a brief explanation of these methods, see Massé, [19].

benefit sequence  $(R_0, R_1, \dots, R_T)$  of a project after discounting each term to the present moment is the *net present value* of the project.

In other words, denoting the weight to be given to economic quantities at  $t=0, 1, \dots, T$  by  $\lambda_0, \lambda_1, \dots, \lambda_T$  respectively, the net present value  $V$  of the project is given by:

$$V = R_0 + \frac{\lambda_1}{\lambda_0} R_1 + \dots + \frac{\lambda_T}{\lambda_0} R_T \quad (\text{I} \cdot 1)$$

This can, as is well known, also be expressed by:

$$V = R_0 + \frac{R_1}{1+r_1} + \dots + \frac{R_T}{(1+r_1)(1+r_2)\dots(1+r_T)} \quad (\text{I} \cdot 2)$$

where  $r_\tau$  ( $\tau=1, \dots, T$ ) denotes the discount rate for the  $\tau$ -th period.

Hence, the relation between the discount rate and the weight is given by:

$$r_\tau = \frac{\lambda_{\tau-1} - \lambda_\tau}{\lambda_\tau} \quad (\tau=1, \dots, T) \quad (\text{I} \cdot 3)$$

Obviously, the net present value of an identical project varies with the set of the discount rates or weights chosen. For the moment we shall proceed with our discussion, taking it for granted that a "correct" set of discount rates or weights are given.

## I. 2. Selection of Project

Suppose that we face a variety of projects, finite in number, the choice of which is our task. Assume that the net present value of each project be known. According to what criteria should we make the selection?

### I. 2. 1. Case without Budgetary Constraint

When there is no budgetary constraint, we can simply base our decision upon the rule: "If the net present value of a project is positive, adopt it. If negative reject it." That this rule accords with the operation of optimization can be seen in the following way.

Now, for the sake of simplicity, let the outlay of the capital cost be made only at the initial moment and denote it by a variable  $K$ . Since the given list of projects provides a technical possibility as a whole, let us construct a function  $G(K)$  by taking the maximum available present value of "benefit minus operating cost" for each value of  $K$ . The function  $V(K) \equiv G(K) - K$  associates the maximum available *net* present value with each value of  $K$ . Assuming  $V(K)$  to be differentiable, the necessary condition for the maximum of  $V(K)$  is given by:

$$\frac{dV}{dK} \equiv \frac{dG}{dK} - 1 = 0^4. \quad (\text{I} \cdot 4)$$

That is, the net present value for the marginal fund should be zero.

4) Marglin, [15], p. 82.

This condition can be approximated in the actual case of finite alternatives by the following rule. Array the projects in decreasing order of the net present value per one unit of capital outlay of each project, and adopt all those that give positive values. (The one with a zero value is a marginal project and it is indifferent whether we adopt it or not.) This rule of thumb corresponds to the maximum condition in the continuous case above and gives the maximum sum of net present values available.

### I. 2. 2. Case with Budgetary Constraint

When there is budgetary constraint, it becomes in general impossible to adopt all the projects with positive net present values, because of the constraint. This can be seen if we consider a non-linear program: "Maximize  $V(K)$  subject to  $K \leq \bar{K}$ ", where  $\bar{K}$  is a certain constant. Denoting the Lagrangean by  $\phi = V(K) + \lambda(\bar{K} - K)$ , the Kuhn-Tucker condition<sup>5)</sup> gives:

$$\begin{aligned} \frac{dV}{dK} &= \lambda && \text{if } K > 0, \\ \frac{dV}{dK} &\leq \lambda && \text{if } K = 0, \\ \lambda &\geq 0 && \text{if } K = \bar{K}, \\ \lambda &= 0 && \text{if } K < \bar{K}. \end{aligned}$$

Therefore, the net present value for the marginal fund should now be  $\lambda$ , which is perhaps positive when the budgetary constraint is effective<sup>6)</sup>.

It seems that this condition could be approximated by the following rule of thumb in actual cases where the number of alternative projects is finite. Array the projects in decreasing order of the net present value per one unit capital outlay of each project, and adopt them one by one until the budget is exhausted. But, in these cases where there are budgetary constraints, if the indivisibility of each project is to be taken into account, the above rule can not in general give the optimal solution. This problem was firstly noticed by Lorie and Savage<sup>7)</sup>, and was attacked and solved by Weingartner by the explicit formulation of an integer program<sup>8)</sup>. The present writer has also attempted to make some points clearer<sup>9)</sup>. But in this paper,

5) Kuhn and Tucker, [12].

6) For the explanation through the classical Lagrange multipliers, see Eckstein, [10], p. 75. For the explanation through non-linear programming, see Marglin, [15], pp. 82-84. Marglin is taking  $m$  projects into consideration there, but one investment opportunity will serve the purpose of this paper for the present. Later in Section III  $n$  projects which are components of the investment opportunity will be taken explicitly into consideration.

7) Lorie, James H., and Savage, Leonald J., "Three Problems in Rationing Capital", *Journal of Business*, XXVIII, 4, Oct. 1955, pp. 229-239. (Reprinted in [20])

8) Weingartner, Martin J., *Mathematical Programming and the Analysis of Capital Budgeting Problems*, 1963.

9) Asanuma, [3] and [4].

we shall go ahead without going into further details about this problem.

### I. 3. The Discount Rate

#### I. 3. 1. Source of the Problem

In cases without budgetary constraints, we can satisfy the optimum condition by merely checking the sign of the net present value of each project. In cases with budgetary constraints, it becomes necessary to handle mathematical programming, though it might be possible to solve simpler programs merely by enumeration. But, in these cases also, the net present value data are indispensable.

However, as already suggested before, to calculate the net present value, the definition and the numerical value of the "correct" set of discount rates should be provided.

In I-2 we proceeded under the assumption that the correct set of discount rates was given. On that account, what was stated there is so general that it is applicable both to public and private investment.

In fact, in a textbook-like world of perfect competition, all economic agents face a single market rate of interest. And, if each agent seeks maximization of his intertemporal utility or the net present value taking this market rate of interest as the discount rate, the market rate of interest tends to be settled at an equilibrium level that brings the aggregate savings and investments of the society to equality. Moreover, this state of equilibrium is a "Pareto optimum" in the sense that no consumer can be made better off as to his intertemporal utility without making someone else worse off. Therefore, in such a world, the market rate of interest can be unique and can have a certain normative implication so that there is no room for any particular problem in the determination of the rate of discount.

However, it has been pointed out on many occasions that there are some problems in the functioning of the market mechanism as to intertemporal resource allocation. Firstly, there has been an issue since Pigou. That is to say, the time preference of an individual for his private decisions is "myopic" in the sense that it tends to overdiscount future consumption. Therefore the rate implied in it does not justify such long-life investments that involve benefits for future generations, which are undertaken frequently by public authorities. Secondly, on account of the imperfections of the capital market and uncertainty about the future the market rate of interest tends to be diversified. Therefore we have no guarantee for its uniqueness as well as its normative property<sup>10)</sup>.

Consequently, to face reality it becomes necessary to reexamine the definition of the optimal discount rate for each type of economic agent

10) About this point, see Kumagai, [13], pp. 288-289.

according to its nature and purposes, and then reflect upon the welfare implications of several discount rates.

Discussions about "the cost of capital" in the field of business finance involve the determination of the optimal discount rate under the imperfections of the market from the viewpoint of a firm. In this paper, however, we shall not go into discussions about the situations specific to business firms; we shall focus our attention on the determination of the discount rate from the viewpoint of public authorities.

### I. 3. 2. Lines of Analysis

It seems, roughly speaking, that there are two different kinds of thinking about the determination of the discount rate for public investment.

The first one proposes to utilize investment efficiency in the field of private investment. The most direct form of this standpoint is to apply the marginal internal rate of return in the private sector without any modification to public investment<sup>11)</sup>. However, as suggested before, there are some doubts in this way of thinking since there exists a gap between the time preference of a typical individual reflected in private investments and the patterns of time distribution of benefits in a variety of public investments even under the assumption of a perfect capital market. There is also a variation of this type of view, which proposes to re-evaluate the investment efficiency achieved by private investment in terms of the same benefit as the aim of public investment to get the marginal social rate of return, and utilize this figure as the discount rate<sup>12)</sup>. Anyway, the first standpoint takes the view of determining the adoption or rejection of public projects exclusively on the basis of their comparison with private marginal projects, instead of forming a positive judgement of value. If the net present value which is acquired by applying the said rate happens to be "positive", it means that the efficiency of the public project in question is superior to that of the private marginal projects, and if "negative", inferior. However, as asserted by Arrow<sup>13)</sup>, though it may be possible to use this type of criterion to answer such a partial question as the relative merits of two different projects, this type of criterion can not answer the question of whether two such investments should both be adopted or not—the problem of the determination of the optimal amount of investment from an over-all point of view. Therefore it may safely be said that the first way of thinking is inadequate for fundamental judgements, although it

11) Marglin, [18], pp. 48-51.

12) *Ibid.*, pp. 51-53.

13) Arrow, [1], pp. 2-3.



may possibly be used with reservation as a convenient tool for making a quick judgement of the efficiency of a relatively small public project.

The second one takes the standpoint of grasping the discount rate as it should be on the basis of information derived from a planner's time preference function which is laid down from an over-all point of view<sup>14</sup>. Needless to say, this standpoint quickly gives rise to the delicate problem of questioning how the planner's time preference function is established. Nevertheless, since we can make use of a plain scheme which has a wide scope of vision as our starting point, we shall develop our further argument along the line of this second standpoint.

## II Marginal Analysis

### II. 1. Investment Decision of an Individual

To see the problem in contrast, let us first see how the investment decision of an individual is analysed by the traditional analysing apparatus. What is going to be taken up here is the method of analysis which originated from Irving Fisher in his "*Theory of Interest*" and has been recently reproduced by Hirshleifer<sup>15</sup>. In this method there appear on the scene at the same time three settings: the time preference of an individual, the productive opportunity of investment, and the market opportunity of investment. Consequently it becomes a characteristic of this method that we can see decisions in the area of production as well as decisions in the area of consumption at a glance.

The essentials of this method of analysis can be grasped geometrically in the simple case of two periods<sup>16</sup>.

The income and consumption for the current period (period 0) is meas-

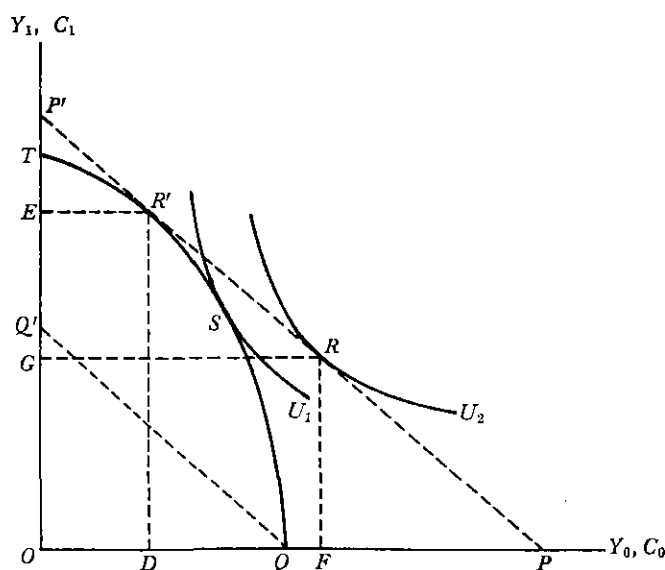


Figure 1

14) Marglin, [18] clearly stands for this view.

15) Hirshleifer, [11].

16) Figure 1 is prepared with some changes in symbols in the diagram on page 206 of Hirshleifer, [11].

ured in a suitable unit (for example in dollars) along the horizontal axis and the income and consumption for the next period (period 1) along the vertical axis. For simplification, the initial position of an individual is represented by  $Q$  (i. e., the income for the current period is  $\bar{Y}_0 = OQ$  and the income for the next period is zero)<sup>17)</sup>.

Now, suppose that he is faced with two kinds of opportunity of investment. One is the productive opportunity for investment, which is represented by the curve  $QR'T$ . For instance, if he makes an investment as in  $QD$  in the current period, he will gain an income of  $OE$  in the next period. Since the investment for the next period can not be taken into consideration in this two-period analysis, the point  $R'$  represents a combination of consumption  $OD$  for the current period and consumption  $OE$  for the next period.

The second opportunity of investment is the money or capital market, where he can lend or borrow at a certain market rate of interest, which is represented by the dotted-lines of the diagram  $QQ'$ ,  $PP'$ , etc. When the capital market is perfect and the interest rate remains constant regardless of the volume of transactions, the market opportunity of investment is represented by a family of parallel straight lines as shown in the diagram. The movement from point  $R'$  to point  $R$  along the market line  $PP'$  means borrowing  $DF$  in the current period with an agreement to return  $EG$  in the next period. Supposing the interest rate is  $i$ , the gradient of the market line is  $-(1+i)$ .

Furthermore it is supposed that this individual has a time preference function  $U(C_0, C_1)$  whose argument  $C_\tau$  is the consumption expenditure of the period  $\tau$  ( $\tau=0, 1$ ). The curves  $U_1$  and  $U_2$  are two of the indifference curves drawn from this time preference function. For instance, every point on  $U_2$  is indifferent to expenditures  $R$  (a combination of consumption expenditures such as  $OF$  for the current period and  $OG$  for the next period) as far as utility for this individual is concerned.

Now, it is assumed that the purpose for this individual is to maximize the utility  $U(C_0, C_1)$  subject to the conditions given above.

We can see from the diagram that the steps needed for climbing to the highest indifference curve attainable starting from point  $Q$  are two. First, the move to point  $R'$  utilizing the productive opportunity of investment. Second, the move to point  $R$  utilizing the market opportunity of investment.  $R$  is the optimal solution and there the individual has increased his utility as much as the difference of  $U_1$  and  $U_2$ , as compared with the point  $S$  where he would be compelled to stop if the market opportunity

17) In fact, a start can be made from any point on the plane of the diagram, as long as the market line through that point intersects the first quadrant. See [11], p. 206.

were not available.

If we note here that the gradient of the market line is  $-(1+i)$ , the equation of the market line being given in the form of  $\frac{Y_1}{1+i} + Y_0 = \text{constant}$ , we can see that the market line has the meaning of an iso-present-value line, when the interest ratio  $i$  is taken to be the discount rate. Correspondingly, the first step mentioned above is no more than an operation to maximize the present value of income under the productive opportunity of investment taking the interest rate as the discount rate.

The productive opportunity of investment  $QR'T$  is drawn as a smooth curve on the assumption that such projects as treated in Section I are available in infinite numbers, each of them being extremely small. If we consider a marginal (productive) investment  $dY_0 (<0)$  at point  $R'$ , since curve  $QR'T$  is a tangent to the market line  $PP'$  at  $R'$ , we have:

$$\frac{dY_1}{dY_0} = -(1+i),$$

and by rearranging, we get:

$$\frac{dY_1}{1+i} + dY_0 = 0.$$

This means that the net present value of the marginal (productive) investment at point  $R'$  is zero. Besides, at points on the right hand side of  $R'$  on the curve  $QR'T$  we have:

$$\frac{dY_1}{dY_0} < -(1+i),$$

so that considering the sign of  $dY_0$ , we have:

$$\frac{dY_1}{1+i} + dY_0 > 0.$$

This means that the net present value of the marginal investment at those points is positive. Lastly, the assumption that  $Q'RT$  is concave to the origin means that investment is assumed to be of decreasing returns.

The foregoing discussion might be sufficient to clarify the correspondence between the practical rule of the net present value method treated in Section I and the Fisher-Hirshleiferian type of marginal analysis.

In passing, note that since the indifference curve  $U_2$  is tangent to the market line at the optimal point  $R$  attained through market transaction, on  $U_2$  we have:

$$\frac{dC_1}{dC_0} = -(1+i),$$

and hence:

$$-\frac{dC_1}{dC_0} - 1 = i.$$

The left side of this equation is none other than what is called the marginal rate of the time preference of this individual.

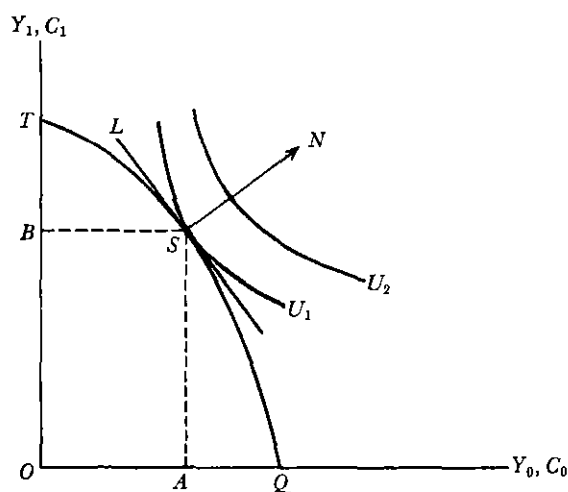


Figure 2

## II. 2. Investment Decision of Planning Authorities

The primary aim of Hirshleifer's analysis in [11] lies in illuminating the meaning and limits of the two practical rules of investment decision—the net present value method and the internal rate of return method—by extending Fisher's method of analysis into cases of imperfect capital market and multi-periods. But here we have no need to trace his arguments thoroughly.

Let us return to the problem of public investment. Disregarding for the moment that a capitalistic economy is in reality a mixture of the public and private sectors, let us suppose that the omniscient planning authorities design an optimal investment on behalf of society as a whole. What changes should take place in the analysis in comparison with the case of private individual investment discussed above?

If we assume a closed economy there is no room for the planning authorities as well as for a Robinson Crusoe to find any market opportunity of investment as a datum. The problem to be solved by the planning authorities is to determine the allocation of consumption and investment so that the utility function of the planning authorities  $U(C_0, C_1)$  can be maximized subject to the productive opportunity  $QST$  for the society as a whole, starting from the initial fund  $OQ$  of the society. The answer will be given by point  $S$  of Figure 2, that is, by the point of  $QST$  tangent to the indifference curve.

In this case, in contrast to the previous case of an individual, the discount rate can not be given ex ante by the market rate of interest as a datum. Rather, when the optimal point  $S$  is determined as a point of  $QST$  tangent to  $U_1$ , the gradient of the tangential line  $L$  gives the discount rate as ex post information. In other words, supposing the gradient of the line  $L$  is  $-(1+r)$ , at the point  $S$  we have:

$$-\frac{dY_1}{dY_0} - 1 = -\frac{dC_1}{dC_0} - 1 = r.$$

Hence the discount rate  $r$  in this case is given by the marginal rate of time preference of the planning authorities at the optimal point.

Now, we have grasped here such a situation as described above as

a characteristic feature of the investment decision of the planning authorities, while on the other hand Hirshleifer has noted that such a situation occurs for an individual in the market mechanism whenever he is obliged to solve a "capital rationing" problem that forces him to make an investment decision without depending on borrowing and within a fixed budget<sup>18)</sup>. And he is of negative opinion with respect to the usefulness of that discount rate which is to be determined in the manner described above, asserting that "..... the present-value or the internal-rate-of-return rules can be formally modified to apply ..... under capital rationing. The discount rate to be used ..... is the rate given by the slope of the ..... tangency; with this rate, the rules give the correct answer. But this rate cannot be discovered until the solution is attained, and so is of no assistance in reaching the solution"<sup>19)</sup>. However, as far as we are starting our argument about public investment criteria in the direction of an active search for an optimal amount of investment as a whole by utilizing the idea of the time preference along the lines suggested by Arrow and Marglin, we can not help facing the above-mentioned situation. Or rather, I propose that by formulating those settings as in Figure 2 explicitly as a mathematical programming problem the discount rate can be grasped expressly as a shadow price, whereby the function to be played by this rate can also be clarified.

### III Non-Linear Programming

Such a type of approach as to formulate "the capital rationing problem" as a programming problem after taking the time preference of consumption into account, like Fisher-Hirshleifer, and then to obtain the discount rate from information provided by the optimal solution has been made by Baumol and Quandt using linear programming<sup>20)</sup>. However, it seems to me better to adopt the formulation of non-linear programming in order to clarify the similarity and difference between the marginal analysis treated in Section II and the mathematical programming. At the same time I introduce a plural number ( $n$ ) of investment projects into the analysis. By doing so it becomes possible not only to relate our discussion to the previous argument in Section I but also to find a way of relaxing in principle the assumption of omniscience on the part of the planning authorities which was imposed on the analysis in Section II.

18) Hirshleifer, [11], p. 213-217.

19) *Ibid.*, p. 216.

20) Baumol and Quandt, [7], pp. 317-329. The problem of public investment is not specifically kept in their mind. Therefore they are rather introducing a preference function of a private investor.

## III. 1. Conditions for an Optimal Solution

We suppose that the problem of the planning authorities is :

$$\text{Maximize } U(C_T, C_{T-1}, \dots, C_1, C_0) \quad (\text{III} \cdot 1)$$

subject to

$$f_i(Y_{i,T}, Y_{i,T-1}, \dots, Y_{i,1}, I_{i0}) \leq 0 \quad (i=1, \dots, n), \quad (\text{III} \cdot 2)$$

$$C_\tau \leq \sum_{i=1}^n Y_{i\tau} \quad (\tau=1, \dots, T), \quad (\text{III} \cdot 3)$$

$$C_0 + \sum_{i=1}^n I_{i0} \leq \bar{Y}_0, \quad (\text{III} \cdot 4)$$

$$C_\tau \geq 0 \quad (\tau=0, 1, \dots, T), \quad (\text{III} \cdot 5)$$

$$Y_{i\tau} \geq 0 \quad (i=1, \dots, n; \tau=1, \dots, T), \quad (\text{III} \cdot 6)$$

$$\text{and } I_{i0} \geq 0 \quad (i=1, \dots, n). \quad (\text{III} \cdot 7)$$

Here the notations are as follows :

$U$  —time preference function of the planning authorities.

$C_\tau$  —consumption outlay for the period  $\tau$ .

$I_{i0}$  —investment outlay for the project  $i$ .

$Y_{i\tau}$  —income produced by the project  $i$  for the period  $\tau$ .

$\bar{Y}_0$  —income for the period 0 given as an initial condition.

$f_i$  —transformation function representing the technical input-output relation for the project  $i$ .

Here it is assumed that the investment outlay is made only for the period 0 and it must be understood that outlay and income are all measured by a certain unit.

When compared with the case of the marginal analysis given in relation with Figure 2 of Section II, not only have we here extended the analysis from a two-period analysis to a multi-period analysis and explicitly introduced  $n$  individual projects which are the components of the investment possibility as a whole, but also (i) inequality constraints are introduced, (ii) the non-negative condition of variables is introduced and (iii) the limitations of resources are explicitly introduced. They, (i), (ii) and (iii), are characteristics of non-linear programming, in contrast to classical marginal analysis<sup>21)</sup>.

Now, assuming that function  $f_i$  is convex (investment projects are technologically decreasing returns) and  $U$  concave (the planner's preference is of decreasing marginal utility), the necessary and sufficient condition for the optimal solution of the problem above is, according to the Kuhn-Tucker Theorem, that there exist non-negative variables  $\lambda_0, \lambda_1, \dots, \lambda_T; \mu_1, \dots, \mu_n$  such that:<sup>22)</sup>

21) For a comparison of classical marginal analysis and non-linear programming, see Chapter 8 of Dorfman, Samuelson and Solow, [9], especially Section 6 (pp. 201-203).

22) Kuhn and Tucker, [12], p. 486.

$$\frac{\partial U}{\partial C_\tau} - \lambda_\tau \left\{ \begin{array}{l} \leq \\ = \end{array} \right\} 0, \quad \text{if } C_\tau \left\{ \begin{array}{l} = \\ > \end{array} \right\} 0 \quad (\tau=0, 1, \dots, T) \quad (\text{III} \cdot 8)$$

$$\lambda_\tau - \mu_i \frac{\partial f_{i\tau}}{\partial Y_{i\tau}} \left\{ \begin{array}{l} \leq \\ = \end{array} \right\} 0, \quad \text{if } Y_{i\tau} \left\{ \begin{array}{l} = \\ > \end{array} \right\} 0 \quad (i=1, \dots, n; \tau=1, \dots, T) \quad (\text{III} \cdot 9)$$

$$-\mu_i \frac{\partial f_i}{\partial I_{i0}} \lambda_0 \left\{ \begin{array}{l} \leq \\ = \end{array} \right\} 0, \quad \text{if } I_{i0} \left\{ \begin{array}{l} = \\ > \end{array} \right\} 0 \quad (i=1, \dots, n) \quad (\text{III} \cdot 10)$$

$$f_i(Y_{i\tau}, Y_{i\tau-1}, \dots, Y_{i1}, I_{i0}) \left\{ \begin{array}{l} \leq \\ = \end{array} \right\} 0, \quad \text{if } \mu_i \left\{ \begin{array}{l} = \\ > \end{array} \right\} 0 \quad (i=1, \dots, n) \quad (\text{III} \cdot 11)$$

$$C_\tau \left\{ \begin{array}{l} \leq \\ = \end{array} \right\} \sum_{i=1}^n Y_{i\tau}, \quad \text{if } \lambda_\tau \left\{ \begin{array}{l} = \\ > \end{array} \right\} 0 \quad (\tau=1, \dots, T) \quad (\text{III} \cdot 12)$$

$$C_0 + \sum_{i=1}^n I_{i0} \left\{ \begin{array}{l} \leq \\ = \end{array} \right\} \bar{Y}_0, \quad \text{if } \lambda_0 \left\{ \begin{array}{l} = \\ > \end{array} \right\} 0 \quad (\text{III} \cdot 13)$$

These  $\lambda_\tau$  ( $\tau=0, 1, \dots, T$ ),  $\mu_i$  ( $i=1, \dots, n$ ) are Lagrange multipliers and they can be interpreted as shadow prices<sup>23</sup>. From (III · 12) and (III · 13) it can be seen that  $\lambda_\tau$  is the shadow price for the income of the period  $\tau$ , and (III · 8) shows that this does not become smaller than the marginal utility of consumption for the period  $\tau$  and that as long as the consumption outlay for the period  $\tau$  is positive, the equality holds.

Also, if we eliminate  $\mu_i$  from (III · 9) and (III · 10), it can be shown that the marginal value product of project  $i$  will not exceed the "price"  $\lambda_0$  of income for the period 0 which is the initial fund of investment, and that the equality holds as long as the investment outlay is actually carried on. Moreover, it is shown that the marginal investment cost of output of project  $i$  will not be lower than the price of output or income for the period  $\tau$  and that the equality holds as long as the output is actually produced.

Furthermore, if (III · 8), (III · 9) and (III · 10) are taken into consideration together, it can be seen that the price  $\lambda_0$  is equal both to the marginal value product of the project actually carried on and to the marginal utility of the consumption for the period 0 if the latter is positive.

On the other hand from (III · 11) it is further shown that project  $i$  is operating on the frontier of its technical possibility if  $\mu_i$  is positive, and operating possibly inwards if  $\mu_i$  is zero. Therefore we may safely say that  $\mu_i$  indicates a social evaluation of the technique of the project  $i$ <sup>24</sup>.

23) As an interesting paper which gives a good example of economic interpretations of Lagrange multipliers of non-linear programming, as shadow prices, see Davis and Whinston, [8], pp. 1-14.

24) Davis and Whinston have given such  $\mu$ 's an interpretation as "the implicit costs of the technology constraints". See [8], p. 4.

Now, once we know that  $\lambda_0, \lambda_1, \dots, \lambda_T$  are evaluations assigned respectively to the incomes for the periods 0, 1, ...,  $T$  from the viewpoint of programming of the planning authorities, we can obtain the discount rate to be applied to the income for the period  $\tau$  by using the following formula which was referred to in Section I:

$$r_\tau = \frac{\lambda_{\tau-1} - \lambda_\tau}{\lambda_\tau} \quad (\text{I} \cdot 3)$$

This is, so to speak, a shadow rate of discount which corresponds to the non-linear program mentioned above.

Here, let us assume that each one of  $n$  projects expressed by the transformation functions (III · 1) is entrusted to a project manager and that the operation of the project is determined by its manager in a decentralized manner under the guide of shadow prices  $\lambda_0, \lambda_1, \dots, \lambda_\tau$  or the shadow rates of discount  $r_1, \dots, r_\tau$ .

Now, suppose that the values of  $\lambda_0, \lambda_1, \dots, \lambda_\tau$  are given. If the manager of project  $i$  is to maximize the net revenue measured by these shadow prices, the problem for him is:

$$\text{Maximize } \sum_{\tau=1}^T \lambda_\tau Y_{i\tau} - \lambda_0 I_{i0} \quad (\text{III} \cdot 14)$$

subject to

$$f_i(Y_{iT}, Y_{iT-1}, \dots, Y_{i1}, I_{i0}) \leq 0, \quad (\text{III} \cdot 15)$$

$$Y_{i\tau} \geq 0, \quad (\tau=1, \dots, T), \quad (\text{III} \cdot 16)$$

$$\text{and } I_{i0} \geq 0. \quad (\text{III} \cdot 17)$$

Since we previously assumed that  $f_i$  is convex, the necessary and sufficient condition for the optimal solution is, again according to the Kuhn-Tucker Theorem, as follows:—

$$\lambda_\tau - \mu_i \frac{\partial f_i}{\partial Y_{i\tau}} \begin{cases} \leq \\ = \end{cases} 0, \quad \text{if } Y_{i\tau} \begin{cases} = \\ > \end{cases} 0 \quad (\text{III} \cdot 18)$$

$$-\lambda_0 - \mu_i \frac{\partial f_i}{\partial I_{i0}} \begin{cases} \leq \\ = \end{cases} 0, \quad \text{if } I_{i0} \begin{cases} = \\ > \end{cases} 0 \quad (\text{III} \cdot 19)$$

$$f_i(Y_{iT}, Y_{iT-1}, \dots, Y_{i1}, I_{i0}) \begin{cases} \leq \\ = \end{cases} 0, \quad \text{if } \mu_i \begin{cases} = \\ > \end{cases} 0. \quad (\text{III} \cdot 20)$$

These conditions are identical with those for the optimal solution of the above-mentioned planning authorities' problem (III · 9), (III · 10) and (III · 11). Consequently, the maximizing behaviour of individual project managers under the guide of shadow prices is compatible with the optimal solution from the over-all viewpoint under the present assumption. However, from (I · 3):

$$\frac{1}{1+r_\tau} = \frac{\lambda_\tau}{\lambda_{\tau-1}}$$

and:

$$\frac{1}{(1+r_1) \cdots (1+r_\tau)} = \frac{\lambda_\tau}{\lambda_0}$$



so that to maximize the net revenue (III · 14) is equivalent to maximize :

$$\frac{Y_1}{1+r_1} + \dots + \frac{Y_T}{(1+r_1)\dots(1+r_T)} - I_0,$$

which is none other than maximizing the net present value of investment. Therefore the proposition above can be restated as follows: the maximization of the net present value of individual projects under the guide of the shadow rates of discount is compatible with the optimal solution from the over-all viewpoint.

### III. 3. Decentralized Computation

So far it has been assumed that shadow prices or shadow rates of discount are given so that the planning authorities are to solve a huge non-linear programming problem collecting all technical information concerning the projects. When projects are large in number, this assumption becomes unreal in all probability, but the results given in III. 2. suggest that there is a way of arriving at the optimal solution without concentrating all information relating to the projects by an iterative procedure between the planning authorities, who announce at each step their tentative prices or discount rates, and the project managers, who behave as profit maximizers taking these prices as guides for their decisions<sup>25)</sup>.

Specifically, the Arrow-Hurwicz method of decentralization based on the gradient method<sup>26)</sup> and the non-linear programming version of the Dantzig-Wolfe decomposition principle<sup>27)</sup> are two representative achievements which we have as guides for further research. By making use of these two guides we can, at least in principle, relax the technical difficulties encountered when we actually try to obtain the rates of discount from nation-wide planning computation.

## IV Distance from Reality

We started by noting that it is questionable to apply the investment efficiency now being achieved by private investment as a criterion for public investment. Then, assuming that there exists a unique value judgement of the society or of the planning authorities with respect to the intertemporal allocation of the consumption outlay, we have tried to derive the optimal discount rate as a shadow price of a problem to determine

25) Marglin in [15], Chapters 4-6, from such a viewpoint has discussed some procedures, i. e., the three algorithms of Lange-Lerner, Arrow-Hurwicz and Zoutendijk-Dorfman. But Marglin's argument in this book is concerned with the maximization of the net present value on the assumption that the discount rate is given.

26) Arrow and Hurwicz, [2].

27) For the decomposition principle for linear programming see Dantzig, G., *Linear Programming and Extensions*, 1963, Chapter 23. For the non-linear programming version, see Malinvaud, [14] and Whinston, [21].

the optimal amount of investment for the society. It must be of course admitted that there exists a considerable distance between here and arriving at a public investment decision in reality.

#### IV. 1. Time Preference Function

The foregoing discussions have assumed that a time preference function of the planning authorities exists. However, in reality, we are obliged to answer an ethical as well as a logical question, such as the possibility of obtaining a rule compatible with democratic procedures. And we must also say that there is little point in discussing it if it can not be found in an operational form.

Marglin [16] attempted to answer the former question.

Private investments of individuals are conducted through the securities market and the interest rate prevailing there reflects their time preferences. On the other hand, the government contemplates the welfare of future generations based on the sacrifices of the present generation in a case like an investment for a dam which lasts for a hundred years, but the profitability of this investment is something quite different from the case of private investment. On what grounds can the government demand such sacrifices of its people?

According to Marglin there are three answers. One of them is the "authoritarian" standpoint which is the view that an individual's time preferences are, if seen from society's viewpoint, myopic and irrational, the contention of which is represented by Pigou. The second answer is the view maintaining that everyone is driven into something like "schizophrenia" because of the discrepancy between his preferences in his capacity as an economic man and his preferences in his capacity as a citizen, the contention of which is represented by Colm. And Marglin himself, taking over the arguments of Baumol and Sen, takes another standpoint which comprises the third answer in trying to find an explanation in the "external effects", which means that each individual's utility depends upon the decision of investment to be made by all other individuals.

We may say that it is rationalized by such an explanation that there should be a difference between the time preference of each individual for his private investment and his time preference for public investment. Then, is it possible to establish the social time preference function on the basis of the time preferences of individuals for public investment? In this connection, however, it has been in fact made clear by Arrow's study on a more general problem that it is impossible to make such an aggregation in a "democratic" way by satisfying the various kinds of desirable axioms. Thus, Marglin after all gives up trying to establish

the social time preference function on the basis of the preferences of individuals.

Then, what about the problem of obtaining an operational form of the time preference function of the planning authorities? According to Marglin it is not necessary to determine the preference function completely, but it is sufficient to know the marginal rate of time preference in the neighbourhood of the optimal investment rate. For this purpose it is required at first to choose a certain optimal growth rate " $g$ ". By using data such as the labour force (and unemployment rate if suitable) it is possible to guess from the optimal growth rate the optimal investment rate " $I$ " and at the same time the optimal discount rate " $r$ ". And, in reality the optimal values for  $g$ ,  $I$  and  $r$  can be obtained by an iterative procedure.

Consequently, as far as the problem of determining the discount rate is concerned, the direction taken by Marglin seems fundamentally similar to the direction discussed in this paper, and it seems that this direction should be linked with the macro-economic optimal growth theory to make further advances and to be more operational.

#### IV. 2. Opportunity Cost

So far my discussion has been carried on as if it were possible in reality for the planning authorities to determine the optimal amount of investment. It may be true to see things in that manner under the conditions in socialist countries, but in capitalist countries there arises another serious problem of finding and realizing a due proportion between public and private investment. Marglin tries to cope with this difficulty mainly by reckoning the opportunity cost as a component of the public investment criteria<sup>28)</sup>, but the analysis of this problem will be taken up at another opportunity<sup>29)</sup>.

#### IV. 3. Dynamics

Investment criteria constitute a problem primarily concerned with the allocation of intertemporal resources, but the framework of the argument given in this paper is not of a dynamic nature. The transformation functions  $f_i$  in the non-linear programming model given in III are fixed up to the future period  $T$ . The outputs produced after period 0 are assumed to be exclusively turned into consumption, and no new invest-

28) Marglin, [17] ; and ditto, [18], pp. 54-67.

29) It is added that, since the problem relating to the composition of benefit and its measurement has been neglected in this paper, the planner's preference function has been considered to be concerned only with the consumption for each period. In reality, particularly in the case of developing countries, there may be some occasions which necessitate the pursuit of several objectives at the same time, such as corrections of the regional inequalities in addition to the mere increase in consumption. Marglin, in [18], with the Indian economy in mind, went into some discussion on this point.

ment opportunity is taken into consideration. However, the majority of discussions about investment criteria that have so far developed, the marginal analysis by Hirshleifer and the linear programming analysis by Baumol-Quandt are of a static nature. A further problem that we must face is a study from the dynamic point of view.

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